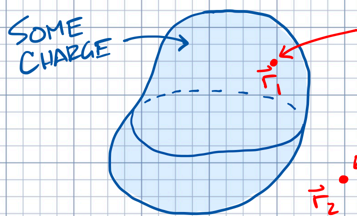
▨ THE ELECTROSTATIC POTENTIAL

- Using the fundamental theorems for divergence & curl, we've learned that:

$$\vec{\nabla}\cdot\vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_0} \qquad \vec{\nabla}\times\vec{E} = 0$$



$\vec{\nabla}\cdot\vec{E}(\vec{r}_1) = \frac{1}{\varepsilon_0}\rho(\vec{r}_1)$

$\vec{\nabla}\times\vec{E}(\vec{r}_1) = 0$

$\vec{\nabla}\cdot\vec{E}(\vec{r}_2) = 0$

$\vec{\nabla}\times\vec{E}(\vec{r}_2) = 0$

The div. of $\vec{E}$ at a point depends on whether there is charge there. But curl of $\vec{E}$ is strictly <u>zero</u>.

- Knowing about the divergence of $\vec{E}$ leads to GAUSS'S LAW. What do we learn from $\vec{\nabla}\times\vec{E} = 0$?

$$\vec{\nabla}\times\vec{E} = 0 \Rightarrow \vec{E} = -\vec{\nabla}V$$

- The fact that $\vec{\nabla}\times\vec{E} = 0$ means we can always find a scalar function V such that $-\vec{\nabla}V$ is the electric field.

- We call V the ELECTROSTATIC POTENTIAL.

- This is a very useful concept; we'll explain why in a moment. But first: <u>Why</u> does $\vec{\nabla}\times\vec{E} = 0$ tell us there must be something like V?

- Suppose you walk into the room & I tell you that it is full of electric field. I give you a little detector that lets you measure $\vec{E}$ at any point.

- Next, I tell you to start @ some point near the door with coordinates $\vec{r}_R = x_R\hat{x} + y_R\hat{y} + z_R\hat{z}$ (the 'R' is for 'reference point') and walk to your desk located @ $\vec{r}_A = x_A\hat{x} + y_A\hat{y} + z_A\hat{z}$. Follow any path you like, but along the way you need to use the detector to monitor $\vec{E}$ and keep a running tally of $-d\vec{\ell}\cdot\vec{E}$. In other words, calculate:

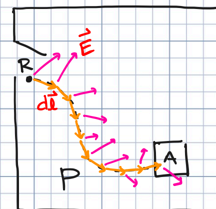
$$-\int_{\vec{r}_R}^{\vec{r}_A} d\vec{\ell}\cdot\vec{E}(\vec{r})$$

Your desk

Path you follow → P

Inf. displacement along the path.

Ref. pt.



- Since $\vec{\nabla}\times\vec{E} = 0$ you get the same result for any path P from the reference point to your desk.

- So the integral depends on $\vec{r}_A$ & $\vec{r}_R$, and also on the details of the electric field along the way but not in a manner where the path matters.

- Let's call this $V_R(\vec{r}_A)$. It depends on where your desk is located — the coordinates $\vec{r}_A$ — and also the ref. point where you started.

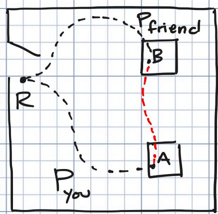
- Meanwhile, your friend was asked to do the same thing, except <u>their</u> desk is located @ $\vec{r}_B = x_B\hat{x} + y_B\hat{y} + z_B\hat{z}$. Their result was $V_R(\vec{r}_B)$.

- Now the two of you <u>compare</u> your results by taking the difference.

$$V_R(\vec{r}_B) - V_R(\vec{r}_A) = -\int_{\vec{r}_R}^{\vec{r}_B} d\vec{\ell}\cdot\vec{E}(\vec{r}) - \left(-\int_{\vec{r}_R}^{\vec{r}_A} d\vec{\ell}\cdot\vec{E}(\vec{r})\right)$$

Could drop this sign if you swap the limits $\vec{r}_A$ & $\vec{r}_R$!

- I then ask "does that <u>really</u> depend on R?" After all, both results were independent of the paths you took (because $\vec{\nabla}\times\vec{E}=0$).



$$V_R(\vec{r}_B) - V_R(\vec{r}_A) = -\int_{\vec{r}_R}^{\vec{r}_B} d\vec{l}\cdot\vec{E} - \int_{\vec{r}_A}^{\vec{r}_R} d\vec{l}\cdot\vec{E}$$
$$= -\int_{\vec{r}_A}^{\vec{r}_B} d\vec{l}\cdot\vec{E}$$

- <u>You agree</u>. After all, when you write out the integrals as above, you realize that it's really just the same as starting @ $\vec{r}_A$, going backwards along P_{you} to $\vec{r}_R$, then from there to $\vec{r}_B$ along path P_{friend}. But $\vec{\nabla}\times\vec{E}=0$, so adding $d\vec{l}\cdot\vec{E}$ from $\vec{r}_A$ to $\vec{r}_B$ should give the same result for <u>any</u> path from A to B.
- You could add up $-d\vec{l}\cdot\vec{E}$ along a path from $\vec{r}_A$ to $\vec{r}_B$ that never visits that point $\vec{r}_R$, so the difference b/t your friend's integral & your integral <u>can not</u> depend on $\vec{r}_R$. In other words:

$$-\int_{\vec{r}_A}^{\vec{r}_B} d\vec{l}\cdot\vec{E}(\vec{r}) = V(\vec{r}_B) - V(\vec{r}_A)$$

- That is, there has to be some function whose value @ $\vec{r}_B$ minus its value @ $\vec{r}_A$ tells you the value of that integral.
- I can move your desks around, too! So this has to be true for any $\vec{r}_A$, any $\vec{r}_B$, and any <u>path</u> from $\vec{r}_A$ to $\vec{r}_B$.

- That's when you remember the FUNDAMENTAL THEOREM FOR GRADIENTS, which says that if you have some function $V(\vec{r})$, then

$$\int_{\vec{r}_A}^{\vec{r}_B} d\vec{l} \cdot \vec{\nabla} V(\vec{r}) = V(\vec{r}_B) - V(\vec{r}_A)$$

for any $\vec{r}_A$, any $\vec{r}_B$, and any path from A to B.

- Comparing your result for $V(\vec{r}_B) - V(\vec{r}_A)$ to the FT for gradients, you have an A-HA! moment:

$$\int_{\vec{r}_A}^{\vec{r}_B} d\vec{l} \cdot \vec{\nabla} V(\vec{r}) = -\int_{\vec{r}_A}^{\vec{r}_B} d\vec{l} \cdot \vec{E}(\vec{r})$$

And the only way this can <u>always</u> be true, for any $\vec{r}_A$ and any $\vec{r}_B$ and any path between them, is if:

$$\Rightarrow \boxed{\vec{E}(\vec{r}) = -\vec{\nabla} V(\vec{r})}$$

- In other words, there has to be a function $V(\vec{r})$ whose (minus) gradient gives $\vec{E}$. We were able to show this with liberal use of path-independence of the integral of $d\vec{l} \cdot \vec{E}$, which follows from $\vec{\nabla} \times \vec{E} = 0$. So it's only true that $\vec{E} = -\vec{\nabla} V$ if $\vec{\nabla} \times \vec{E} = 0$!

- The function $V(\vec{r})$ is <u>not</u> <u>unique</u>. There's nothing tricky here: if I add a <u>constant</u> to $V(\vec{r})$ that doesn't change $\vec{E}$, because the gradient of a constant is zero! $\vec{\nabla}(V(\vec{r}) + \text{const}) = \vec{\nabla} V(\vec{r})$.

- This fact is sitting right there in our integral for $V_R(\vec{r})$:

$$V_R(\vec{r}) = -\int_{\vec{r}_R}^{\vec{r}} d\vec{l}' \cdot \vec{E}(\vec{r}')$$

If we move the reference pt. to $\vec{r}_{\tilde{R}}$ this changes the potential:

$$V_{\tilde{R}}(\vec{r}) = -\int_{\vec{r}_{\tilde{R}}}^{\vec{r}} d\vec{l}' \cdot \vec{E}(\vec{r}')$$

- But the result can't depend on the path from $\vec{r}_{\tilde{R}}$ to $\vec{r}$, and we're free to evaluate it along a path from $\vec{r}_{\tilde{R}}$ to $\vec{r}_R$ to $\vec{r}$!

$$V_{\tilde{R}}(\vec{r}) = \underbrace{-\int_{\vec{r}_R}^{\vec{r}} d\vec{l}' \cdot \vec{E}(\vec{r}')}_{\text{What we called } V_R(\vec{r})} \underbrace{- \int_{\vec{r}_{\tilde{R}}}^{\vec{r}_R} d\vec{l}' \cdot \vec{E}(\vec{r}')}_{}$$

Depends on $\vec{r}_R$ & $\vec{r}_{\tilde{R}}$, but not $\vec{r}$.

- This is just another way of saying that 'the' potential @ $\vec{r}$ is always given with respect to some reference point, and if we change the reference point then the potential @ $\vec{r}$ changes by a constant.

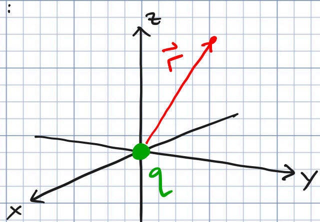
- So, given an electric field $\vec{E}(\vec{r})$, the electrostatic potential @ $\vec{r}$ relative to some reference point $\vec{r}_R$ is:

$$V(\vec{r}) = -\int_{\vec{r}_R}^{\vec{r}} d\vec{l} \cdot \vec{E}(\vec{r})$$

- You will see an example of how to compute $V(\vec{r})$ starting from $\vec{E}(\vec{r})$ in class.

- But this can't be it, right? The potential only makes things useful (in the sense of letting us avoid Coulomb integrals) if we can compute it directly from what we know about the charges, and then <u>use it</u> to find $\vec{E}(\vec{r})$!

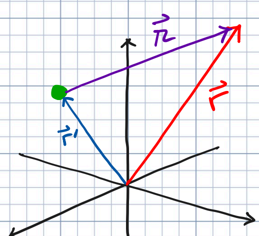
- How do we do this? In Phys 126 you learned that the potential for a point charge q @ the origin is:



$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{r}$$

A <u>SCALAR</u>!

- Or if you move the origin, so that q is located @ $\vec{r}\,'$, this is:



$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{\mathscr{r}}|}$$
$$= \frac{1}{4\pi\varepsilon_0}\frac{q}{|\vec{r}-\vec{r}\,'|}$$

- And if there are multiple charges q_i located @ position $\vec{r}_i\,'$, $i=1,\dots,N$, then the potential @ $\vec{r}$ is:

$$V(\vec{r}) = \frac{1}{4\pi\varepsilon_0}\sum_{i=1}^{N}\frac{q_i}{|\vec{r}-\vec{r}_i\,'|}$$

- From here, we make the jump to a <u>distribution</u> of charge, w/ infinitesimal bits $dq(\vec{r}\,')$ located along a curve ($dq(\vec{r}\,') = \lambda(\vec{r}\,')\,dl'$), spread out over a surface ($dq(\vec{r}\,') = \sigma(\vec{r}\,')\,da'$), or throughout a volume ($dq(\vec{r}\,') = \rho(\vec{r}\,')\,d\tau'$). Then:

$$V(\vec{r}) = \begin{cases} \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int_P dl' \,\frac{\lambda(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \\[2ex] \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int_S da' \,\frac{\sigma(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \\[2ex] \dfrac{1}{4\pi\varepsilon_0}\displaystyle\int_V d\tau' \,\frac{\rho(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} \end{cases}$$

<u>IMPORTANT</u>: Arrived at this starting from V for a collection of point charges, which have $V(r\to\infty)\to 0$. These formulas can break down for things like an '∞ sheet' or '∞ line' of charge — we'll discuss in class!

- Finding $V(\vec{r})$ is usually preferable to evaluating the corresponding Coulomb integral for $\vec{E}(\vec{r})$ because:

  (i) V is a scalar & $\vec{E}$ is a vector. You do one integral instead of 3.

  (ii) The integrals for V are sometimes easier.

  (iii) $\vec{\nabla}\cdot\vec{E}(\vec{r}) = \frac{1}{\varepsilon_0}\rho(\vec{r}) \Rightarrow \nabla^2 V(\vec{r}) = -\frac{1}{\varepsilon_0}\rho(\vec{r})$, and we can use powerful techniques for solving diff. eqns. to find V this way.

- In class you will see an example of calculating V by setting up & evaluating the integrals above.